Exercise 7.2.16

(a) Solve Example 7.2.1, assuming that the parachute opens when the parachutist's velocity has reached $v_i = 60$ mi/h (regard this time as $t = 0$). Find $v(t)$.

(b) For a skydiver in free fall use the friction coefficient $b = 0.25$ kg/m and mass $m = 70$ kg. What is the limiting velocity in this case?

Solution

Example 7.2.1 in the text applies Newton's second law to determine the equation of motion for a parachutist falling straight down subject to the force of gravity and a quadratic drag force.

$$\sum \mathbf{F} = m\mathbf{a}$$

Only the motion in the y-direction is relevant.

$$\sum F_y = ma_y$$

The positive y-direction is chosen to point downward, so the forces on the left side will be mg and $-bv_y^2$. Let $v_y = v$ and $a_y = dv_y/dt = dv/dt$.

$$mg - bv^2 = m\frac{dv}{dt}$$

Even though this ODE is nonlinear, it can be solved by separating variables.

$$\frac{dv}{mg - bv^2} = \frac{dt}{m}$$

Integrate both sides.

$$\int^v \frac{dx}{mg - bx^2} = \int^t \frac{ds}{m} + C_1$$

Evaluate the integral on the right, and make the substitution,

$$x = \sqrt{\frac{mg}{b}}\sin\theta \quad \rightarrow \quad x^2 = \frac{mg}{b}\sin^2\theta \quad \Rightarrow \quad mg - bx^2 = mg\cos^2\theta$$
$$dx = \sqrt{\frac{mg}{b}}\cos\theta\, d\theta,$$

in the integral on the left.

$$\int^{\sin^{-1}\left(v\sqrt{\frac{b}{mg}}\right)} \frac{\sqrt{\frac{mg}{b}}\cos\theta\, d\theta}{mg\cos^2\theta} = \frac{t}{m} + C_1$$

Bring the constant in front and write $1/\cos\theta$ as $\sec\theta$.

$$\frac{1}{\sqrt{mgb}}\int^{\sin^{-1}\left(v\sqrt{\frac{b}{mg}}\right)} \sec\theta\, d\theta = \frac{t}{m} + C_1$$

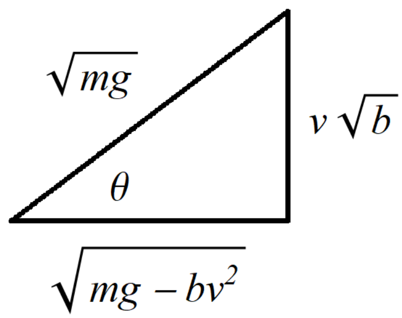
Evaluate the integral.

$$\frac{1}{\sqrt{mgb}}\left(\ln|\sec\theta+\tan\theta|\right)\Big|^{\sin^{-1}\left(v\sqrt{\frac{b}{mg}}\right)} = \frac{t}{m}+C_1$$

Substitute the limit of integration.

$$\frac{1}{\sqrt{mgb}}\ln\left|\sec\left[\sin^{-1}\left(v\sqrt{\frac{b}{mg}}\right)\right]+\tan\left[\sin^{-1}\left(v\sqrt{\frac{b}{mg}}\right)\right]\right| = \frac{t}{m}+C_1 \tag{1}$$

Draw the implied right triangle to determine the secant and tangent of inverse sine.



Here in the figure $\theta = \sin^{-1}(v\sqrt{b/mg})$, so

$$\sec\theta = \frac{\sqrt{mg}}{\sqrt{mg-bv^2}} \quad \text{and} \quad \tan\theta = \frac{v\sqrt{b}}{\sqrt{mg-bv^2}}.$$

Consequently, equation (1) becomes

$$\frac{1}{\sqrt{mgb}}\ln\left|\frac{\sqrt{mg}}{\sqrt{mg-bv^2}}+\frac{v\sqrt{b}}{\sqrt{mg-bv^2}}\right| = \frac{t}{m}+C_1.$$

Combine the two fractions.

$$\frac{1}{\sqrt{mgb}}\ln\left|\frac{\sqrt{mg}+v\sqrt{b}}{\sqrt{mg-bv^2}}\right| = \frac{t}{m}+C_1$$

Multiply both sides by $\sqrt{mgb}$, using a new constant C_2 for $C_1\sqrt{mgb}$.

$$\ln\left|\frac{\sqrt{mg}+v\sqrt{b}}{\sqrt{mg-bv^2}}\right| = \sqrt{\frac{gb}{m}}t+C_2$$

Exponentiate both sides.

$$\begin{aligned}\left|\frac{\sqrt{mg}+v\sqrt{b}}{\sqrt{mg-bv^2}}\right| &= e^{t\sqrt{gb/m}+C_2} \\ &= e^{t\sqrt{gb/m}}e^{C_2}\end{aligned}$$

Remove the absolute value sign on the left by placing $\pm$ on the right.

$$\frac{\sqrt{mg}+v\sqrt{b}}{\sqrt{mg-bv^2}} = \pm e^{C_2}e^{t\sqrt{gb/m}}$$

Use a new constant A for $\pm e^{C_2}$.

$$\frac{\sqrt{mg}+v\sqrt{b}}{\sqrt{mg-bv^2}} = Ae^{t\sqrt{gb/m}} \tag{2}$$

Apply the initial condition $v(0) = v_0$ here to determine A.

$$\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}} = A$$

Square both sides of equation (2) and proceed to bring all terms to the left side.

$$\frac{mg+2v\sqrt{mgb}+v^2b}{mg-bv^2} = A^2e^{2t\sqrt{gb/m}}$$

$$mg+2v\sqrt{mgb}+v^2b = mgA^2e^{2t\sqrt{gb/m}} - bv^2A^2e^{2t\sqrt{gb/m}}$$

$$v^2\left(b+bA^2e^{2t\sqrt{gb/m}}\right)+v\left(2\sqrt{mgb}\right)+\left(mg-mgA^2e^{2t\sqrt{gb/m}}\right)=0$$

Use the quadratic formula to solve for v.

$$v = \frac{-2\sqrt{mgb}\pm\sqrt{\left(2\sqrt{mgb}\right)^2-4\left(b+bA^2e^{2t\sqrt{gb/m}}\right)\left(mg-mgA^2e^{2t\sqrt{gb/m}}\right)}}{2\left(b+bA^2e^{2t\sqrt{gb/m}}\right)}$$

Because the parachutist is falling down in the positive y-direction, the velocity is positive, which means the positive sign is chosen.

$$\begin{aligned}
v(t) &= \frac{-2\sqrt{mgb}+\sqrt{4mgb-4mgb\left(1+A^2e^{2t\sqrt{gb/m}}\right)\left(1-A^2e^{2t\sqrt{gb/m}}\right)}}{2b\left(1+A^2e^{2t\sqrt{gb/m}}\right)} \\
&= \frac{-2\sqrt{mgb}+\sqrt{4mgb-4mgb\left(1-A^4e^{4t\sqrt{gb/m}}\right)}}{2b\left(1+A^2e^{2t\sqrt{gb/m}}\right)} \\
&= \frac{-2\sqrt{mgb}+\sqrt{4mgbA^4e^{4t\sqrt{gb/m}}}}{2b\left(1+A^2e^{2t\sqrt{gb/m}}\right)} \\
&= \frac{-2\sqrt{mgb}+2\sqrt{mgb}A^2e^{2t\sqrt{gb/m}}}{2b\left(1+A^2e^{2t\sqrt{gb/m}}\right)} \\
&= \sqrt{\frac{mg}{b}}\left(\frac{-1+A^2e^{2t\sqrt{gb/m}}}{1+A^2e^{2t\sqrt{gb/m}}}\right)
\end{aligned}$$

Note that if $v_0 = 0$, then $A = 1$, and the right side can be written in terms of hyperbolic tangent using the identity,

$$\tanh\alpha X = \frac{e^{2\alpha X}-1}{e^{2\alpha X}+1}.$$

Substitute the formula for A to obtain the general formula for the parachutist's velocity.

$$\boxed{v(t) = \sqrt{\frac{mg}{b}}\left[\frac{-1 + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2 e^{2t\sqrt{gb/m}}}{1 + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2 e^{2t\sqrt{gb/m}}}\right]}$$

If $v_0 = 60$ mi/h, then

$$v(t) = \sqrt{\frac{mg}{b}}\left[\frac{-1 + \left(\frac{\sqrt{mg}+60\sqrt{b}}{\sqrt{mg-3600b}}\right)^2 e^{2t\sqrt{gb/m}}}{1 + \left(\frac{\sqrt{mg}+60\sqrt{b}}{\sqrt{mg-3600b}}\right)^2 e^{2t\sqrt{gb/m}}}\right].$$

Rewrite the boxed formula so that the exponents of e are negative.

$$\begin{aligned} v(t) &= \sqrt{\frac{mg}{b}}\left[\frac{-1 + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2 e^{2t\sqrt{gb/m}}}{1 + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2 e^{2t\sqrt{gb/m}}}\right] \cdot \frac{e^{-2t\sqrt{gb/m}}}{e^{-2t\sqrt{gb/m}}} \\ &= \sqrt{\frac{mg}{b}}\left[\frac{-e^{-2t\sqrt{gb/m}} + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2}{e^{-2t\sqrt{gb/m}} + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2}\right] \end{aligned}$$

Now take the limit of $v(t)$ as $t \to \infty$.

$$\begin{aligned} \lim_{t\to\infty} v(t) &= \lim_{t\to\infty} \sqrt{\frac{mg}{b}}\left[\frac{-e^{-2t\sqrt{gb/m}} + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2}{e^{-2t\sqrt{gb/m}} + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2}\right] \\ &= \sqrt{\frac{mg}{b}}\left[\frac{0 + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2}{0 + \left(\frac{\sqrt{mg}+v_0\sqrt{b}}{\sqrt{mg-bv_0^2}}\right)^2}\right] \\ &= \sqrt{\frac{mg}{b}}(1) \\ &= \sqrt{\frac{mg}{b}} \end{aligned}$$

Regardless of what initial velocity the parachutist has, the same limiting velocity will be reached. If $m = 70$ kg, $g = 9.81$ m/s^2, and $b = 0.25$ kg/m, then

$$\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{b}} = \sqrt{\frac{(70 \cancel{\text{kg}})\left(9.81 \frac{\text{m}}{\text{s}^2}\right)}{0.25 \frac{\cancel{\text{kg}}}{\text{m}}}} \approx 50 \frac{\text{m}}{\text{s}}.$$